Stochastic quantization and holographic Wilsonian renormalization group

Jae-Hyuk Oh[a1] and Dileep P. Jatkar[a2]

Harish-Chandra Research Institute,
Chhatnag Road, Jhunsi, Allahabad-211019, India[a]

Abstract

We study relation between stochastic quantization and holographic Wilsonian renormalization group flow. Considering stochastic quantization of the boundary on-shell actions with the Dirichlet boundary condition for certain AdS bulk gravity theories, we find that the radial flows of double trace deformations in the boundary effective actions are completely captured by stochastic time evolution with identification of the AdS radial coordinate 'r' with the stochastic time 't' as $r = t$. More precisely, we investigate Langevin dynamics and find an exact relation between radial flow of the double trace couplings and 2-point correlation functions in stochastic quantization. We also show that the radial evolution of double trace deformations in the boundary effective action and the stochastic time evolution of the Fokker-Planck action are the same. We demonstrate this relation with a couple of examples: (minimally coupled)massless scalar fields in AdS_2 and $U(1)$ vector fields in AdS_4.

[1]e-mail:jack.jaehyuk.oh@gmail.com
[2]e-mail:dileep@hri.res.in

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1 Introduction

AdS/CFT correspondence has shed a lot of light on strongly coupled field theories. Investigation of the holographic renormalization group (RG) flows, for example, has become a useful tool to understand the Wilsonian RG flow of strongly coupled dual field theories. In fact, with the recent improved understanding of the holographic RG[1, 2], it has become clear that these two approaches to RG flow of the boundary theory are consistent with each other[3, 4]. In the dual theory defined in AdS space, AdS radial coordinate r is identified with Wilsonian RG-direction, in other words, the radial direction is related to the energy scale of the CFT. AdS boundary($r = 0$) is treated as the UV-region whereas Poincaré horizon($r = \infty$) is treated as the IR-region. For finite nonvanishing values of r, one can define a CFT at the intermediate energy scale.

A method for computing holographic Wilsonian RG flows of certain deformations of the theory defined on the UV boundary was developed in [3, 4]. (For earlier work on relevance of multi-trace operators to holographic RG, see [5].) The flow equation has a form of the Hamilton-Jacobi equation in the limit when bulk action is restricted to the terms up to two derivatives. The most important feature of this computation is that even though one has a free theory in the dual gravity, the flow equations necessarily contain double trace deformations as long as non-zero momenta along boundary directions are turned on. For zero momentum case,

the flow becomes rather trivial without these double trace couplings. The double trace deformation coupling has evoked a lot of interest recently. For example, for double trace couplings of transverse(longitudinal) boundary $U(1)$ gauge fields appearing in the boundary effective actions with bulk $U(1)$ gauge fields in AdS_4, the equations of these couplings correspond to the flow equations of transverse(longitudinal) conductivities in the dual fluid system defined on AdS boundary [3].

The double trace couplings show several fixed points in UV boundary, which depend on the boundary conditions on it. In [4], they provide examples of the flow equations for bulk scalar fields with its mass m, which is in the range that $-\frac{d^2}{4} \leq m^2 \leq -\frac{d^2}{4}+1$, where d is spacetime dimension of the boundary and for $U(1)$ gauge fields in AdS_4. The most important property of both bulk theories is that they allow alternative quantization[12, 13, 14, 15, 18, 19]. In this case, one can impose both Neumann boundary condition as well as Dirichlet boundary condition on the conformal boundary. These boundary conditions correspond to alternative and standard quantization respectively and they lead to different UV fixed points. Moreover, there are many classes of flows which do not start from fixed points in the UV region.

However, in the IR region, near Poincaré horizon, it turns out that most of the flows converge to a single fixed point for these examples[4](where background geometry of the bulk is Poincaré patch of the pure AdS space). Moreover, it turns out that the boundary effective action in IR region has the same form as the classical effective action Γ. Γ is derived from the on-shell action I_{os} by Legendre transform, where the on-shell action is obtained from bulk action by imposing Dirichlet boundary condition at the UV boundary.

There have been some attempts in the past trying to relate AdS/CFT and stochastic quantization[22, 23, 24, 28]. Stochastic quantization[16, 17] is a quantization method for Euclidean field theories where one starts with a d-dimensional Euclidean action, $S_c(\phi)$(which is also called the classical action). The coupling of the field ϕ to the surrounding is mimicked by Gaussian white noise η, which is the source of randomness or stochasticity in the system. Stochastic system evolves along the stochastic time t, which is different from the Euclidean time, τ. At very late time $t \to \infty$, the system settle down to an equilibrium state, and partition function of it provides correlation functions of quantum field theory with action S_c. Even if the system starts with d-dimensional Euclidean action, the resulting theory is $d+1$-dimensional since even in equilibrium the system is evolving along the stochastic time 't'. In fact, AdS/CFT correspondence has similar structure. Conformal field theories on the d-dimensional AdS boundary are related to $d+1$-dimensional bulk string theories and the radial coordinate 'r' in AdS space has similar role to play as the stochastic time 't'.

In particular, there is a rather concrete conjecture for the relation[24], which basically depends on an identification of a partition function derived from the holographic method with the stochastic partition function. Holographic partition function is given by

$$Z_{hol} = \int_{\phi(r=0)=\phi_0} [D\phi] e^{-S_{bulk}(\phi)} = e^{-W(\phi_0)}, \tag{1.1}$$

[3]There are many computations of transport coefficients using holographic Wilsonian RG(equivalently the sliding membrane paradigm), such as shear viscosity [6, 7] and conductivities[8, 9, 10, 11].

[4]This is no longer true when the bulk geometry is that of an extremal black brane. In that case, there is emergent 1-dimensional CFT near the black brane horizon and couplings of bulk fields admitting alternative quantization even in AdS_2, may give rise to other nontrivial fixed points.

where the boundary is AdS boundary which is located at $r = 0$, ϕ denotes the bulk field and ϕ_0 is its boundary value. In the above expression, we have imposed Dirichlet boundary condition at $r = 0$ and $W(\phi_0)$ is called generating functional since ϕ_0 becomes a source term which couples to a composite operator in the dual CFT. Another partition function Z' was constructed[24] from Z_{hol},

$$Z' = \int [D\phi_0] e^{W(\phi_0)} Z_{hol}, \tag{1.2}$$

which is a partition function with a new non-trivial weight, $e^{W(\phi_0)}$. On the other hand, stochastic partition function is spelled out as

$$Z_{SQ} = \int [D\phi_0] e^{-\frac{S_c(\phi_0)}{2}} [D\phi] e^{-S_{FP}(\phi)}, \tag{1.3}$$

where S_{FP} is called Fokker-Planck action and $S_c(\phi_0)$ is classical action which appears in the boundary at $t = 0$, where the stochastic time evolution starts from $-\infty$ and ends up with $t = 0$, i.e. $-\infty < t < 0$. Fokker-Planck action can be made out of the classical action by promoting the boundary field $\phi_0 \to \phi_0(t)$ [5].

After identifying the two different partition functions, Z' and Z_{SQ}, it was conjectured[24] that there is a one to one correspondence between the Fokker-Planck action S_{FP} and the classical action S_c in stochastic partition function, and the bulk action S_{bulk} and the generating functional W in holographic partition function respectively, provided that the stochastic time '$-t$' is identified to the radial coordinate 'r'.

In fact, in [20, 21, 25, 26], the authors have studied conformally coupled scalar field theory in AdS_4 and obtained a boundary on-shell action at the conformal boundary. The boundary action becomes scalar field theory action with two derivative kinetic term and 6-point self interacting vertex by truncation up to leading order in large conformal coupling expansion. It follows from their proposal that $S_c = -2W(\phi)$, and one can then construct the Fokker-Planck action and which reproduces the bulk action (again) by truncation up to leading order in large coupling expansion of boundary ϕ^6 interaction.

These two independent results motivated us to study relation between the holographic renormalization group and the stochastic quantization. The main motivation is that if such an identification can reproduce the Fokker-Planck action using boundary on-shell action, then one may be able to reconstruct radial evolution of the boundary effective action via stochastic time evolution using this Fokker-Planck action.

In this paper, we have developed a one to one correspondence between these two schemes, the Holographic Wilsonian Renormalization Group and the Stochastic quantization, by analyzing their Hamiltonian formalism for scalar fields and abelian gauge fields such that their dynamics in the AdS_4 space is reproduced in the limit of two derivative bulk actions. While the Holographic Wilsonian Renormalization Group is closely tied with the AdS geometry, the Hamiltonian formalism for Stochastic processes has no a priori relation with AdS/CFT. As will be explained in Sec.2.1, the Hamiltonian formalism is suitable for both of holographic renormalization group and stochastic quantization. For holographic renormalization group, it

[5] For a detailed discussion, see Sec.2.1.

is given by

$$\partial_\epsilon \psi_H(\phi, r) = -\int_{r=\epsilon} d^d x \mathcal{H}_{RG}(-\frac{\delta}{\delta\phi}, \phi)\psi_H(\phi, r), \tag{1.4}$$

where $\mathcal{H}_{RG}$ is obtained from the bulk action by Legendre transform and $\psi_H = e^{-S_B}$, where S_B is boundary deformation(boundary effective action). The stochastic Hamiltonian formalism, on the other hand gives

$$\partial_t \psi_S(\phi, t) = -\int d^d x \mathcal{H}_{FP}(\frac{\delta}{\delta\phi}, \phi)\psi_S(\phi, t), \tag{1.5}$$

where $\mathcal{H}_{FP}$ is called the Fokker-Planck Hamiltonian, which is related to the Fokker-Planck action by Legendre transform. The wave function $\psi_S(\phi, t)$ is given by

$$\psi_S(\phi, t) = P(\phi, t) e^{\frac{S_c(\phi(t))}{2}}, \tag{1.6}$$

where $P(\phi, t)$ is called the probability distribution, which provide non-trivial weight for the stochastic partition function.

We focus on the similarity between them, and developed one to one correspondence of quantities appearing in each Hamiltonian dynamics. We briefly discuss our proposal here. It is easy to note by comparing these two Hamiltonian dynamics that, **(1) the stochastic time** t **should be identified to the radial coordinate** r, which is a statement similar to that in[24], but this time, **precisely** $r = t$. We also assume that in the absence of any deformation terms at the boundary of the AdS space **(2) the classical action,** S_c **and the on-shell action,** $I_{os}(\phi)$ **(or classical effective action,** $\Gamma(\phi)$ **through Legendre transform from it) are related as** $S_c \equiv 2\Gamma(\phi) = -2I_{os}(\phi)$**.** Finally, we identify these two different Hamiltonians as **(3)** $\mathcal{H}_{RG}(r) = \mathcal{H}_{FP}(t)$, which is consistent with proposal (1).

We will discuss our proposal in detail in Sec.2.2. Here we would like to summarize the reason for proposing the identification(2). As mentioned in the discussion of the holographic renormalization in pure AdS, there is a single IR fixed point for most of the curves(flows) and the IR effective action has the same form as the classical effective action Γ on the conformal boundary. Similar phenomenon happens in the case of stochastic quantization. One starts with a system described by a classical action S_c. Under stochastic time evolution the system will settle in an equilibrium state which can be described in terms of the Euclidean partition function with an action which provides quantization of S_c at some late time t. Therefore, if we impose identification (2) then we are, at least, guaranteed that most of the IR behavior of the holographic renormalization group flow and the late time behavior of stochastic time evolution are the same. UV behavior, as we will see, turns out to be dependent on the initial condition for the stochastic time evolution. We will discuss appropriate choice of initial condition in Sec.3.

Another point that we would like to mention here is related to the conjecture (3). This is a non-trivial statement since the form of Fokker-Planck Hamiltonian density is completely determined by S_c. Therefore, conjecture (3) completely depends on the proposal (2) and it could be a conditional statement. However, we believe that if certain specific choice of S_c gives rise to correct IR behavior(equivalently, late time behavior), then that S_c will provide a correct(similar for the weak condition) form of the Fokker-Planck Hamiltonian.

We have obtained the following relations as a consequence of our proposal. Firstly, we have found that double trace deformation part of the boundary effective action, **(1) S_B is given by**

$$S_B = \int_{t_0}^{t} d\tilde{t} \int d^d x \mathcal{L}_{FP}(\phi(t,x)), \tag{1.7}$$

in the classical limit, which is the main result of this paper. Secondly, we have studied the Langevin dynamics to establish **(2) the relation between stochastic 2-point correlation functions and the double trace coupling in AdS/CFT**

$$< \phi_q(t)\phi_{q'}(t) >_H^{-1} = < \phi_q(t)\phi_{q'}(t) >_S^{-1} - \frac{1}{2}\frac{\delta^2 S_c}{\delta\phi_q(t)\delta\phi_{q'}(t)}, \tag{1.8}$$

where

$$< \phi_q(t)\phi_{q'}(t) >_H^{-1} = \frac{\delta^2 S_B}{\delta\phi_q(t)\delta\phi_{q'}(t)}, \tag{1.9}$$

the coefficient of double trace deformation term and $< \phi_q(t)\phi_{q'}(t) >_S$ is stochastic 2-point correlation function.

To test our proposal, we have worked out two examples, which are (minimally coupled)massless scalar fields in AdS_2 and $U(1)$ gauge field theory in AdS_4. It turns out that stochastic quantization successfully captures the radial evolution of double trace couplings appearing holographic renormalization group computations of these examples through the above two relations.

These two models presents several interesting features. Firstly, they allow alternative quantization. Secondly, their actions are Weyl invariant. The first condition provides a good playground for analyzing a variety of boundary conditions, which means the model will provide more than one fixed point on the UV boundary and diverse radial flows. The second condition will make computations easy because Weyl invariance implies there will be no divergent behavior of the bulk modes and as a result the counter-term action is not necessary. Another merit of the second condition is that bulk action will effectively defined on the flat space(See beginning of Sec.3 for details).

Finally, it turns out that the Fokker-Planck Hamiltonian(Lagrangian) density obtained from such a classical action, $S_c = 2\Gamma$ approximately reconstructs the form of the bulk Hamiltonian(Lagrangian) density, therefore conjecture (3) is partially proved in these cases. For the massless scalar field case, the bulk Lagrangian is completely reconstructed. However, The $U(1)$ gauge fields case is not since to evaluate boundary on-shell action we have chosen a gauge. Therefore, the bulk Lagrangian is recovered up to gauge degrees of freedom.

2 $AdS/$(free)CFT and Stochastic Quantization

2.1 Stochastic Quantization and Holographic Wilsonian Renormalization Group

In this section, we will discuss similarity between holographic Wilsonian renormalization group flows($HWRG$)[3, 4] and stochastic quantization(SQ)[16, 17]. We will set up one to one map-

ping between various quantities such as the two-point correlators, boundary effective actions and so on, appearing in the $HWRG$ and those in the SQ.

2.1.1 Holographic Wilsonian Renormalization Group

In this subsection, we briefly review the $HWRG$. We start with a bulk action in the Euclidean AdS_{d+1} as

$$S = \int_{r>\epsilon} dr d^d x \sqrt{g} \mathcal{L}(\phi, \partial\phi) + S_B, \tag{2.1}$$

where ϵ is an arbitrary cut-off in the radial direction. The background AdS metric is given by

$$ds^2 = \frac{dr^2 + \sum_{i=1}^d dx_i dx_i}{r^2}, \tag{2.2}$$

and S_B is interpreted as the boundary effective action.

From the condition that variation of the full action S vanishes, one can define the canonical momentum Π_ϕ:

$$\Pi_\phi = \sqrt{g}\frac{\partial \mathcal{L}}{\partial(\partial_r \phi)} = \frac{\delta S_B}{\delta \phi(x)}, \tag{2.3}$$

as a boundary condition. Since the cut-off ϵ in the action(2.1) is arbitrary, the physical requirement that the total action(2.1) does not depend on the cut-off ϵ gives rise to the following equation:

$$\partial_\epsilon S_B = -\int_{r=\epsilon} d^d x \left(\frac{\delta S_B}{\delta \phi} \partial_r \phi - \mathcal{L}(\phi, \partial\phi) \right) = \int_{r=\epsilon} d^d x \mathcal{H}_{RG}(\frac{\delta S_B}{\delta \phi}, \phi), \tag{2.4}$$

where for the second equality in (2.4), we have performed Legendre transform from the Lagrangian density, $\mathcal{L}$, using the definition of canonical momentum Π_ϕ to $\mathcal{H}_{RG}$ which is the Hamiltonian density. The eq.(2.4) is, in fact, semi-classical version of the Schrödinger type equation. To see this more precisely, one can define the wave functional ψ as

$$\psi_H = \exp(-S_B), \tag{2.5}$$

and the Schrödinger type wave equation is

$$\partial_\epsilon \psi_H = -\int_{r=\epsilon} d^d x \mathcal{H}_{RG}(-\frac{\delta}{\delta\phi}, \phi)\psi_H. \tag{2.6}$$

In this discussion, we have implicitly assumed that the Hamiltonian density is quadratic in canonical momentum. Eq.(2.4) is recovered in the semi-classical limit, *i.e.*, $\left(\frac{\delta S_B}{\delta \phi}\right)^2 >> \frac{\delta^2 S_B}{\delta \phi^2}$ and ignoring terms proportional to $\frac{\delta^2 S_B}{\delta\phi^2}$.

2.1.2 Stochastic Quantization

The Hamiltonian description of a system in terms of fictitious stochastic time 't' is defined in the stochastic quantization[6] as well. We will now briefly discuss the method of stochastic quantization, for which we mostly follow [16]. The basic notion of stochastic quantization comes from the similarity between partition function of Euclidean field theory and partition function of a statistical system in equilibrium. The Euclidean N-point correlation function is given by

$$< \phi(x_1)...\phi(x_N) >= \int D\phi \frac{e^{-\frac{1}{\hbar}S_c(\phi)}}{\int D\tilde{\phi}e^{-\frac{1}{\hbar}S_c(\tilde{\phi})}}\phi(x_1)...\phi(x_N), \tag{2.7}$$

where S_c is an Euclidean action(It is also called the 'classical action'). However, once we identify $\hbar \equiv k_B T$, where k_B is Boltzmann constant and T is temperature, this partition function can also be interpreted as the partition function of a statistical system in equilibrium with a bath at temperature T. Stochastic process describes evolution of a statistical system from a non-equilibrium configuration, along a fictitious time to an equilibrium configuration at the very late time. The fictitious time here is called the stochastic time and it is different from Euclidean time $x_0 \equiv \tau$. Unlike in the equilibrium state, the measure in eq.(2.7) for non-equilibrium states is not a Boltzmann distribution. Therefore, we define correlation functions in non-equilibrium states with a general measure $P(\phi, t)$(which is called the probability distribution)as

$$< \phi(x_1)...\phi(x_N) >= \int D\phi P(\phi, t)\phi(x_1)...\phi(x_N). \tag{2.8}$$

Technically, stochastic process is describing stochastic time evolution of $P(\phi, t)$ and once $P(\phi, t)$ is known, then the correlation functions during stochastic precess are entirely known.

The Langevin Dynamics The first realization of this idea was given by Parisi and Wu[29]. To understand their treatment, let us consider $\phi(x)$ which is a scalar field in d-dimensional space with a classical action, S_c. We suppose that this field $\phi(x)$ interacts with an imaginary thermal reservoir with temperature T and the system evolves, by interacting with this thermal reservoir, along the fictitious stochastic time t. Since the system is evolving with time t, we promote the field $\phi(x)$, for it to be time dependent, to

$$\phi(x) \to \phi(x, t) \tag{2.9}$$

and we expect that in large t limit the system approaches a state of thermal equilibrium state.

It turns out that the relaxation process satisfies the following equation of motion:

$$\frac{\partial \phi(x,t)}{\partial t} = -\frac{1}{2}\frac{\delta S_c}{\delta \phi(x,t)} + \eta(x,t), \tag{2.10}$$

which is called the Langevin equation, where $\eta(x, t)$ is the Gaussian white noise, which provides interactions with thermal reservoir. This white noise has Gaussian probability distribution and

[6] For reviews, see [16, 17].

its expectation values are defined as

$$< \eta(x_1,t_1)...\eta(x_N,t_N) >= \frac{\int D\eta(x,t)\eta(x_1,t_1)...\eta(x_N,t_N)e^{-\frac{1}{2}\int d^dxdt\eta^2(x,t)}}{\int D\eta(x,t)e^{-\frac{1}{2}\int d^dxdt\eta^2(x,t)}}. \tag{2.11}$$

Explicit computations of these correlation functions provide rules for the correlations of $\eta(x,t)$ namely

$$\begin{aligned} < \eta_{i,q}(t) > &= 0\ ,\ < \eta_{i,q}(t)\eta_{j,q'}(t') >= \delta_{ij}\delta^d(q-q')\delta(t-t'), \qquad (2.12)\\ < \eta_{i_1,q_1}(t_1)...\eta_{i_{2k},q_{2k}}(t_{2k}) > &= \sum_{\text{all possible pairs of i and j}} \Pi_{\text{pairs}} < \eta_{i,q_i}(t_i)\eta_{j,q_j}(t_j) >, \end{aligned}$$

and any correlations with odd number of insertions of $\eta_{i,q}(t)$ vanish.

Finally, to obtain correlation functions of $\phi(x,t)$, we need to solve the Langevin equation and get solution of $\phi(x,t)$ with explicit dependence on $\eta(x,t)$, then we get

$$< \phi(x_1,t_1)...\phi(x_N,t_N) >= \frac{\int D\eta(x,t)\phi(x_1,t_1)...\phi(x_N,t_N)e^{-\frac{1}{2}\int d^dxdt\eta^2(x,t)}}{\int D\eta(x,t)e^{-\frac{1}{2}\int d^dxdt\eta^2(x,t)}}. \tag{2.13}$$

Obtaining the probability distribution from the Langevin dynamics As we mentioned, getting probability distribution $P(\phi,t)$ is very crucial for stochastic process. Let us get into the details for this. The partition function for Langevin dynamics is

$$Z = \int D\eta(x,t)e^{-\frac{1}{2}\int d^dxdt\eta^2(x,t)}. \tag{2.14}$$

To get more useful information from the partition function, it is convenient to switch from $\eta(x,t)$ to $\phi(x,t)$ in the partition function by using the Langevin equation(2.10),

$$Z = \int D\phi(x,t)det\left(\frac{\delta\eta}{\delta\phi}\right)P(\phi,t_0)exp\left[-\frac{1}{2}\int_{t_0}^{t} d^dxd\tilde{t}\left(\dot{\phi}(x,\tilde{t})+\frac{1}{2}\frac{\delta S_c}{\delta\phi(x,\tilde{t})}\right)^2\right], \tag{2.15}$$

where

$$P(\phi,t_0) = \Pi_x\delta^d\left(\phi(x,t_0)-\phi_0(x)\right), \tag{2.16}$$

which gives the initial condition for $\phi(x)$, t_0 is initial time and 'dot' denotes derivative with respect to $\tilde{t}$. The Jacobian factor can be written more explicitly using the Langevin equation as,

$$det\left(\frac{\delta\eta}{\delta\phi}\right) = \exp\left[\frac{1}{4}\int_{t_0}^{t} d\tilde{t}\int d^dx\frac{\delta^2 S_c}{\delta\phi^2(x,\tilde{t})}\right]. \tag{2.17}$$

Once we expand the exponent of (2.15), it gives a total derivative term with respect to $\tilde{t}$. This total derivative term provides boundary contribution at $\tilde{t}=t_0$ and $\tilde{t}=t$. With all this taken into account we get

$$Z = \int D\phi(x,t_0)P(\phi,t_0)e^{\frac{S_c(\phi(t_0))}{2}}D\phi(x,t)e^{-\frac{S_c(\phi(t))}{2}}[D\phi]\exp\left(-\int_{t_0}^{t} d\tilde{t}\int d^dx\mathcal{L}_{FP}(\phi(\tilde{t},x))\right), \tag{2.18}$$

where

$$[D\phi] = \Pi_{t_0<\tilde{t}<t} D\phi(x,\tilde{t}) \tag{2.19}$$

and

$$\mathcal{L}_{FP} = \frac{1}{2}\left(\frac{\partial\phi(x)}{\partial t}\right)^2 + \frac{1}{8}\left(\frac{\delta S_c}{\delta\phi(x)}\right)^2 - \frac{1}{4}\frac{\delta^2 S_c}{\delta\phi^2(x)}, \tag{2.20}$$

which is called the Fokker-Planck Lagrangian density. From this expression, N-point correlation functions can be easily computed. By comparison this with (2.8), one can write the probability distribution function as

$$P(\phi,t) = \exp\left[-\frac{S_c(\phi(t))}{2} - \int_{t_0}^{t} d\tilde{t}\int d^d x \mathcal{L}_{FP}(\phi(\tilde{t},x))\right]. \tag{2.21}$$

The Fokker-Planck Approach The equation satisfied by the probability distribution $P(\phi)$ can be derived using the Langevin equation,

$$\frac{\partial P(\phi,t)}{\partial t} = \frac{1}{2}\int d^d x \frac{\delta}{\delta\phi(x,t)}\left(\frac{\delta S_c}{\delta\phi(x,t)} + \frac{\delta}{\delta\phi(x,t)}\right)P(\phi,t). \tag{2.22}$$

We will express this equation in a more suggestive form by defining a wave function ψ_S as

$$\psi_S(\phi,t) \equiv P(\phi,t)e^{\frac{S_c}{2}}, \tag{2.23}$$

and demanding that this wave function satisfies the Schrödinger type equation of motion:

$$\partial_t\psi_S(\phi,t) = -\int d^d x \mathcal{H}_{FP}(\frac{\delta}{\delta\phi},\phi)\psi_S(\phi,t), \tag{2.24}$$

where $\mathcal{H}_{FP}$ is called the Fokker-Planck Hamiltonian, which is given by

$$\begin{aligned} \mathcal{H}_{FP} &\equiv \frac{1}{2}\left(-\frac{\delta}{\delta\phi(x)} + \frac{1}{2}\frac{\delta S_c}{\delta\phi(x)}\right)\left(\frac{\delta}{\delta\phi(x)} + \frac{1}{2}\frac{\delta S_c}{\delta\phi(x)}\right) \\ &= -\frac{1}{2}\frac{\delta^2}{\delta\phi^2(x)} + \frac{1}{8}\left(\frac{\delta S_c}{\delta\phi(x)}\right)^2 - \frac{1}{4}\frac{\delta^2 S_c}{\delta\phi^2(x)} \end{aligned} \tag{2.25}$$

In fact, the Fokker-Planck Lagrangian (2.20) is related to $\mathcal{H}_{FP}$ through Legendre transform.

2.2 Relations between Stochastic Quantization and Holographic Wilsonian Renormalization Group

The Fokker-Planck approach In [24], it was suggested that some quantities in the stochastic quantization may be identified with quantities appearing in AdS/CFT in the following manner

- The fictitious stochastic time, 't' $\rightarrow$ AdS radial coordinate 'r' from its boundary to the interior,

- The Fokker-Planck action: $S_{FP} \rightarrow$ The bulk action: $S_{bulk}[\phi_I(r)]$,
- The classical action, $S_{cl} \rightarrow -2I_{os}[\phi_I^{(0)}] = 2\Gamma[\phi_I^{(0)}]$,

where I_{os} is the bulk on-shell action, Γ is the classical effective action, the index, I, denotes any index that the bulk fields(we suppress this index in the most of the following discussion), ϕ_I carry and $\phi_I^{(0)}$ denotes the boundary value of the bulk field on the conformal boundary. In this section, we will investigate how many of these assumptions are valid and if they are all valid, then what kind of information in AdS/CFT is reproduced by using stochastic quantization. More precisely, we will figure out a one to one mapping between quantities appearing in the stochastic quantization and the Holographic Wilsonian renormalization group.

We start with a comparison between (2.6) and (2.24). They look very similar, and in fact, they will be the same if the following two conditions are satisfied:

- Condition 1: Stochastic time 't' is identified to radial coordinate 'r' in AdS space.
- Condition 2: The Fokker-Planck Hamiltonian, $\mathcal{H}_{FP}$ has the same form(or similar form as a weak condition) as the Hamiltonian of holographic renormalization group flow, $\mathcal{H}_{RG}$. The exact relation is given by

$$\mathcal{H}_{FP}(t) = \mathcal{H}_{RG}(r) \;\; \text{provided} \;\; r = t. \tag{2.26}$$

The condition 1 is similar to the first suggestion of [24], listed above. However, the condition 2 is rather non-trivial. It is hard to see if the two Hamiltonian densities are the same or not. Since the form of Fokker-Planck Hamiltonian highly depends on the classical action S_c, determination of S_c is therefore very crucial. To determine S_c, we follow the suggestion of [24] namely

$$S_c = 2\Gamma(\phi), \tag{2.27}$$

and we demand that *this form of classical action reproduces the same(or similar for a mild condition) relation between $\mathcal{H}_{RG}$ and the Fokker-Planck Hamiltonian $\mathcal{H}_{FP}$.* If this condition is satisfied then the Fokker-Planck Lagrangian density can be derived by Legendre transform and the second condition from [24] will also be satisfied. We therefore propose that $S_c = 2\Gamma(\phi)$ gives the correct choice of the classical action S_c.

Under these conditions, the two Hamiltonian equations of motion (the Fokker-Planck and the Renormalization Group) are identified. As a consequence of this, the two wave functions ψ_H and ψ_S will also be identified as

$$\psi_H = e^{-S_B} \equiv \psi_S = P(\phi, t) e^{\frac{S_c}{2}}. \tag{2.28}$$

In the classical limit, by using the expression of the probability distribution (2.21), we can write S_B explicitly in terms of $\mathcal{L}_{FP}$ as

$$S_B = \int_{t_0}^{t} d\tilde{t} \int d^d x \mathcal{L}_{FP}(\phi(\tilde{t}, x)). \tag{2.29}$$

In the limit of $t \to \infty$(the same with $r \to \infty$), $P(\phi, t = \infty)$ is expected to become the Boltzmann distribution and in that limit, S_B will become

$$e^{-S_B} = e^{-S_c + \frac{S_c}{2}} = e^{-\frac{S_c}{2}} = e^{-\Gamma(\phi)}. \tag{2.30}$$

Therefore, at the very late time, S_B converges to $\Gamma(\phi)$, which is consistent with the IR effective action from the Holographic Wilsonian renormalization group flows.

Langevin approach Equation (2.28) also provides a relation between deformation couplings in the holographic effective action and correlation functions in stochastic quantization. For a simple case, we assume that the theory that we are dealing with is a free theory, so only two point correlators are non-trivial. From the definition of stochastic correlations(2.8), the two point function is given by

$$< \phi_{q_1}(t_1)\phi_{q_2}(t_2) >_S = \int D\phi e^{-S_P(t)} \phi_{q_1}(t_1)\phi_{q_2}(t_2), \tag{2.31}$$

where we define a new quantity S_p as $P(\phi, t) \equiv e^{-S_P(t)}$. Since we have assumed that this is a free theory, $S_P(t)$ will have the form

$$S_P(t) = \frac{1}{2} \int \mathcal{K}_q(t)\phi_q(t)\phi_{-q}(t) d^d q, \tag{2.32}$$

where $\mathcal{K}_q(t)$ is the kernel and q is the d-dimensional momentum. From this definition, the two point (equal time) correlation function in stochastic quantization is

$$< \phi_{q_1}(t)\phi_{q_2}(t) >_S = \frac{1}{\mathcal{K}_q(t)} \delta^d(q_1 + q_2). \tag{2.33}$$

Notice that in AdS/CFT, double trace couplings in holographic effective action have a slightly different definition. According to the relation(2.28), $S_B = S_P - \frac{S_c}{2}$ in the limit of free theory, we define a kernel of the double trace operator in holographic effective action as

$$< \phi_q(r)\phi_{q'}(r) >_H^{-1} = \frac{\delta^2 S_B}{\delta\phi_q(r)\delta\phi_{q'}(r)}. \tag{2.34}$$

From the relation (2.28), we have

$$< \phi_q(r)\phi_{q'}(r) >_H = \frac{1}{\mathcal{K}_q(r) - \tilde{k}_q(r)} \delta^d(q + q'). \tag{2.35}$$

$\tilde{k}_q(r)$ is the kernel of S_c, we have defined S_c as

$$S_c = \int \tilde{k}_q(r)\phi_q(r)\phi_{-q}(r) d^d q, \tag{2.36}$$

and the kernel $\tilde{k}_q(r)$ is formally given by,

$$k_q(r)\delta^d(q + q') = \frac{1}{2} \frac{\delta^2 S_c}{\delta\phi_q(r)\delta\phi_{q'}(r)}. \tag{2.37}$$

Therefore, by comparing (2.33) with (2.35) and using (2.37), we conclude that there is a relation between two point correlators on both sides as

$$< \phi_{q_1}(t)\phi_{q_2}(t) >_H^{-1} = < \phi_{q_1}(t)\phi_{q_2}(t) >_S^{-1} - \frac{1}{2}\frac{\delta^2 S_c}{\delta\phi_q(t)\delta\phi_{-q}(t)}, \tag{2.38}$$

where we identify r to t and δ-function in the momentum space is ignored in this relation.

3 Examples

3.1 The simplest example, massless scalar fields in AdS_2

We start with a very simple model, (minimally coupled)massless scalar field(or zero form field) in Euclidean AdS_2. The action is given by

$$S_{bulk} = \frac{1}{2}\int dr d\tau \sqrt{g} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{3.1}$$

where $g_{\mu\nu}$ is AdS_2 metric and g is its determinant. AdS_2 metric is given by

$$ds^2 = \frac{dr^2 + d\tau^2}{r^2}, \tag{3.2}$$

where r is radial coordinate in AdS with $0 \leq r \leq \infty$, $r = 0$ is AdS boundary and $r = \infty$ is Poincaré horizon. τ is Euclidean time(We reserve t to denote the stochastic time.).

Although this is very simple example, it has several merits. First of all, this action is Weyl invariant. The Weyl invariance is manifest since the background metric has a form of

$$g_{\mu\nu} = \frac{\delta_{\mu\nu}}{r^2}, \tag{3.3}$$

and substituting this metric into the action(3.1), we get

$$S_{bulk} = \frac{1}{2}\int_{\mathbb{R}^2_+} dr d\tau \partial_\mu\phi\partial_\mu\phi, \tag{3.4}$$

where the space-time indices are contracted by $\delta^{\mu\nu}$, which is the Kronecker δ and $\mathbb{R}^2_+$ denotes, say, the space corresponding to the upper half of $\mathbb{R}^2$, since the coordinate 'r' is semi-infinite. Another feature is that due to this Weyl invariance, there are no divergent terms in the bulk action as $r \to 0$. Therefore, no counter term action is necessary.

Secondly, this action allows 'alternative quantization' for its boundary CFT. It is well-known that in AdS/CFT, for a particular range of mass square of bulk scalar fields, $-\frac{d^2}{4} \leq m^2 \leq -\frac{d^2}{4} + 1$, there are two possible quantizations. Here d is dimension of boundary space-time. Each quantization scheme depends on the boundary condition of the bulk field, which is either Dirichlet or Neumann boundary condition. For AdS_2 case, $d = 1$ and the mass square range is given by $-\frac{1}{4} \leq m^2 \leq \frac{3}{4}$. Therefore, massless scalar fields admits 'alternative quantization'.

3.1.1 Bulk Solutions and their Boundary On-shell Actions

Holographic Boundary On-shell Action In this section, we apply standard AdS/CFT techniques to our model and find out its boundary on-shell action. To obtain this, we will solve bulk system in the limit of Einstein gravity. In a given AdS background, we get an equation of motion of the scalar field as

$$0 = (\partial_r^2 + \partial_\tau^2)\phi(r,\tau). \tag{3.5}$$

We will solve this equation in the momentum space by using Fourier transform,

$$\phi(r,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \phi_\omega(r) e^{-i\omega\tau}. \tag{3.6}$$

Then, the equation of motion(3.5) becomes

$$0 = (\partial_r^2 - \omega^2)\phi_\omega(r). \tag{3.7}$$

The most general form of the bulk solution is given by

$$\phi_\omega(r) = \phi_\omega^{(0)} \cosh(|\omega|r) + \frac{\phi_\omega^{(1)}}{|\omega|} \sinh(|\omega|r), \tag{3.8}$$

where $\phi_\omega^{(0)}$ and $\phi_\omega^{(1)}$ are arbitrary frequency dependent functions. Another condition that we need to consider is the regularity of the solution on the Poincaré horizon. The solution(3.8) is exponentially growing as in the interior and is divergent at $r = \infty$. To prevent such a behavior, we set

$$\phi^{(0)} + \frac{\phi^{(1)}}{|\omega|} = 0. \tag{3.9}$$

Then, the solution becomes

$$\phi_\omega(r) = \phi_\omega^{(0)} e^{-|\omega|r}. \tag{3.10}$$

Substituting this solution back in the action, up to equation of motion, we get

$$S_{bulk} = \lim_{\epsilon\to 0} \frac{1}{2} \int_{r=\epsilon} d\omega \phi_\omega(r)\partial_r\phi_{-\omega}(r) = -\frac{1}{2}\int d\omega |\omega| \phi_\omega^{(0)}\phi_{-\omega}^{(0)}, \tag{3.11}$$

where we have used boundary expansion of $\phi_\omega(r)$ as

$$\phi_\omega(r\to 0) = \phi_\omega^{(0)} - |\omega|\phi_\omega^{(0)} r + O(r^2). \tag{3.12}$$

On the boundary of AdS, we impose the Dirichlet boundary condition, $\delta\phi = 0$. In this case, there is no deformation term need to be added to the bulk action. Therefore, the bulk action itself becomes the on-shell action, $I_{os}(\phi) = S_{bulk}(\phi)$. The boundary value of $\phi(r)$ is then interpreted as a source term which couples to a composite operator in boundary CFT. Thus we can then identify the on-shell action with the generating functional with source ϕ as $I_{os}(\phi) = W(\phi)$.

To obtain classical effective action Γ, we define canonical momentum as

$$\Pi_{\phi,\omega} = \frac{\partial \mathcal{L}_{bulk}}{\partial \phi'_\omega} = \phi'_{-\omega}(r) = -|\omega|\phi_{-\omega}, \tag{3.13}$$

where 'prime' denotes derivative with respect to r. The classical effective action is defined by Legendre transform of the generating functional as

$$\Gamma[\phi] = -\Pi_{\phi,\omega}\phi_\omega + W[\phi]. \tag{3.14}$$

Then, we get [7]

$$\Gamma[\phi] = -W[\phi] = \frac{1}{2}\int d\omega|\omega|\phi^{(0)}_\omega\phi^{(0)}_{-\omega}. \tag{3.16}$$

Zero frequency solution and its boundary on-shell action In the limit of $\omega = 0$, the bulk equation of motion (3.7) is given by

$$\partial_r^2\phi = 0, \tag{3.17}$$

and its most general solution is

$$\phi = \phi^{(0)} + \phi^{(1)}r. \tag{3.18}$$

When we impose regularity condition on the solution in the interior(at $r = \infty$), we are forced to set $\phi^{(1)} = 0$. Now, to get boundary on-shell action, we substitute (3.18) into the expression of on-shell action(3.11). This gives $I_{os}(\phi) = 0$, because the regular solution satisfies $\partial_r\phi = 0$. This means that canonical momentum of ϕ is also zero as $\Pi = \partial_r\phi = 0$. By Legendre transform, we get its classical effective action which is zero, $\Gamma(\phi) = 0$, too.

3.1.2 Holographic Wilsonian renormalization group

We start with (2.4) and our two dimensional bulk Lagrangian(3.4). Substitution (3.4) into (2.4) provides holographic Hamilton-Jacobi equation as

$$\partial_\epsilon S_B = -\frac{1}{2}\int_{r=\epsilon} d\omega\left(\left(\frac{\delta S_B}{\delta\phi_\omega}\right)\left(\frac{\delta S_B}{\delta\phi_{-\omega}}\right) - \omega^2\phi_\omega\phi_{-\omega}\right). \tag{3.19}$$

Let us solve this equation by assumption of the form of S_B as

$$S_B = \Lambda(\epsilon) + \int\frac{d\omega}{2\pi}\sqrt{\gamma(\epsilon)}\mathcal{J}(\epsilon,\omega)\phi_{-\omega} - \frac{1}{2}\int\frac{d\omega}{2\pi}\sqrt{\gamma(\epsilon)}\mathcal{F}(\epsilon,\omega)\phi_\omega\phi_{-\omega}, \tag{3.20}$$

[7] In fact, we need to express the classical effective action in terms of Π, which is given by

$$\Gamma[\Pi] = \frac{1}{2(2\pi)}\int\frac{d\omega}{|\omega|}\Pi_\omega(\tilde{r})\Pi_{-\omega}(\tilde{r}), \tag{3.15}$$

where Π is vacuum expectation value when one imposes Dirichlet boundary condition. However, we express this in terms of ϕ, since it is more convenient for the later discussion.

where $\Lambda(\epsilon)$, $\mathcal{J}(\epsilon,\omega)$ and $\mathcal{F}(\epsilon,\omega)$ are unknown functions of radial cut-off ϵ, and especially $\mathcal{F}(\epsilon,\omega)$ is interpreted as double trace coupling. $\gamma(\epsilon)$ is determinant of (one dimensional) induced metric at $r=\epsilon$ hypersurface, in fact, it is given by $\gamma = \frac{g(\epsilon)}{g_{rr}(\epsilon)} = \frac{1}{\epsilon^2}$. Putting the ansatz (3.20) into (3.19) and comparing the coefficients of field ϕ_ω, we get the following three equations:

$$\begin{aligned}
\partial_\epsilon \Lambda(\epsilon) &= -\frac{1}{2}\int_\epsilon \frac{d\omega}{(2\pi)^2} J(\epsilon,\omega)J(\epsilon,-\omega), & (3.21)\\
\partial_\epsilon J(\epsilon,-\omega) &= \frac{1}{2\pi} J(\epsilon,\omega) f(\epsilon,-\omega), & (3.22)\\
\partial_\epsilon f(\epsilon,\omega) &= \frac{1}{2\pi} f(\epsilon,-\omega) f(\epsilon,\omega) - 2\pi\omega^2, & (3.23)
\end{aligned}$$

where $J(\epsilon,\omega) \equiv \sqrt{\gamma(\epsilon)}\mathcal{J}(\epsilon,\omega)$ and $f(\epsilon,\omega) \equiv \sqrt{\gamma(\epsilon)}\mathcal{F}(\epsilon,\omega)$. We can then plug the definition of $f(\epsilon,\omega)$ into (3.23) to obtain a equation in terms of double trace coupling, $\mathcal{F}$ as

$$r\partial_r \mathcal{F}(r,\omega) = \mathcal{F}(r,\omega) + \frac{1}{2\pi}\mathcal{F}(r,\omega)\mathcal{F}(r,-\omega) - 2\pi\omega^2 r^2. \tag{3.24}$$

The Hamiltonian equation of motion of the bulk field ϕ_ω given by

$$\Pi_\omega = \partial_r \phi_{-\omega}, \quad \text{and} \quad \partial_r \Pi_\omega = \omega^2 \phi_{-\omega}, \tag{3.25}$$

can be used to seek the solutions of $\Lambda(\epsilon)$, $J(\epsilon,\omega)$ and $f(\epsilon,\omega)$. They are

$$\begin{aligned}
f(\epsilon,\omega) &= -2\pi\frac{\Pi_\omega(\epsilon)}{\phi_{-\omega}(\epsilon)}, \quad J(\epsilon,\omega) = -\frac{\beta_\omega}{\phi_\omega(\epsilon)}, & (3.26)\\
\text{and} \quad \partial_\epsilon \Lambda(\epsilon) &= -\frac{1}{2}\int_{r=\epsilon} \frac{d\omega}{(2\pi)^2}\frac{\beta_\omega \beta_{-\omega}}{\phi_\omega(\epsilon)\phi_{-\omega}(\epsilon)},
\end{aligned}$$

where β_ω is an arbitrary frequency dependent function.

Zero frequency solution Now, let us evaluate the effective action S_B by using the above solution. Solution of (3.23) in $\omega = 0$ limit is given by

$$f(r) = -2\pi\frac{\Pi}{\phi} = -\frac{2\pi\chi}{1+\chi r}, \tag{3.27}$$

where ϕ is a linear combination of independent solutions

$$\phi = A\phi_1 + B\phi_2, \text{ where } \phi_1 = 1 \text{ and } \phi_2 = r, \tag{3.28}$$

and A, B are arbitrary constants and $\chi \equiv \frac{B}{A}$. Using this solution, we can write the expression for the double trace coupling $\mathcal{F}$ as

$$\mathcal{F} = -\frac{2\pi\chi r}{1+\chi r}. \tag{3.29}$$

It is easy to see that eq.(3.29) has two different fixed points, $\mathcal{F} = 0$ and $\mathcal{F} = -2\pi$(These fixed points are solutions of eq.(3.24)). Another point to note is that, in the IR region, we have a

single fixed point, $\mathcal{F} = -2\pi$ if $\chi \neq 0$, therefore almost every flows will end up with that fixed point. If $\chi = 0$, then in the IR region $\mathcal{F} = 0$ is a fixed point. In the UV region also we have these two different fixed points but their properties are different. The $\mathcal{F} = -2\pi$ is a fixed point if and only if $\chi = \pm\infty$, whereas $\mathcal{F} = 0$ is a fixed point when $\chi = 0$[4].

From the above solution, we can obtain S_B as

$$S_B = \frac{1}{2}\frac{\chi}{1+\chi r}\phi^2, \tag{3.30}$$

where we have evaluated double trace coupling only(We will deal with double trace couplings only in the most of the following discussion.) and integration over frequency is removed because we are at $\omega = 0$(Effectively, we have inserted $\delta(\omega)$ in the integrand). In Sec.3.1.3, one will see that (3.30) is precisely reproduced by stochastic quantization.

Solution with non-zero frequency The solutions of bulk equations of motion, (3.25) are linear combination of $\cosh(|\omega|r)$ and $\sinh(|\omega|r)$ when frequency is turned on. Using this fact, the effective action S_B is given by

$$S_B(r) = \frac{1}{2}\int d\omega|\omega|\left(\frac{\sinh(|\omega|r) + \tilde{\phi}_\omega\cosh(|\omega|r)}{\cosh(|\omega|r) + \tilde{\phi}_\omega\sinh(|\omega|r)}\right)\phi_\omega\phi_{-\omega}, \tag{3.31}$$

where $\tilde{\phi}_\omega$ is an frequency dependent real function[8]. As $r \to \infty$, the boundary action approaches its IR region in the sense of holographic renormalization group. The form of IR effective action is

$$S_B(r=\infty) = \frac{1}{2}\int d\omega|\omega|\phi_\omega\phi_{-\omega}, \tag{3.32}$$

unless $\tilde{\phi}_\omega = -1$. If $\tilde{\phi}_\omega = -1$, then

$$S_B(r) = -\frac{1}{2}\int d\omega|\omega|\phi_\omega\phi_{-\omega}. \tag{3.33}$$

UV and IR fixed points of the double trace coupling and its flows It is clear that there are several UV fixed points for the double trace coupling, $\mathcal{F}$. In the UV-region($r \to 0$), that there are two fixed points as $\mathcal{F}(r,\omega) = 0$ and $\mathcal{F}(r,\omega) = -2\pi$, since the last term in (3.24) vanishes. Classification of these fixed points depends on boundary conditions, *i.e.*, on the choice of $\tilde{\phi}_\omega$. If we choose $\tilde{\phi}_\omega = \pm\infty$, we have $\mathcal{F}(r,\omega) = -2\pi$ fixed point and for $\tilde{\phi}_\omega = 0$, we have $\mathcal{F}(r,\omega) = 0$ fixed point. However, it is not certain if $\mathcal{F}(r,\omega)$ has IR fixed points from (3.24) since the last term in it cannot be ignored in large r region anymore. In fact, from the solution of double trace coupling,

$$\mathcal{F}(r,\omega) = -2\pi|\omega|r\frac{\sinh(|\omega|r) + \tilde{\phi}_\omega\cosh(|\omega|r)}{\cosh(|\omega|r) + \tilde{\phi}_\omega\sinh(|\omega|r)}, \tag{3.34}$$

one can recognize that it converges to a single fixed point, $\mathcal{F}(r,\omega) = -\infty$ in IR region for any values of $\tilde{\phi}_\omega$ except $\tilde{\phi}_\omega = -1$. If $\tilde{\phi}_\omega = -1$, $\mathcal{F}(r,\omega) = \infty$ is IR fixed point.

[8] If $\tilde{\phi}_\omega$ is not real, then the double trace deformation is not hermitian.

3.1.3 Stochastic quantization of the classical effective action: Zero frequency

As per our proposal, relation between S_{cl} and AdS/CFT is that $S_{cl} = 2\Gamma[\phi]$, where $\Gamma[\phi]$ is classical effective action on AdS_2 boundary for the massless scalar field. Leaving out this connection, we will not use any information from AdS/CFT for our computations in this section, we will only use stochastic quantization techniques.

The Fokker-Planck action Let us evaluate stochastic time evolution of the system in which the classical action is given by $S_c = 2\Gamma(\phi)$ as conjectured in Sec.2.2. As we discussed in Sec.3.1.1, in the case of zero frequency, the classical effective action, $\Gamma(\phi) = 0$. From the expression of Fokker-Planck action, we have

$$S_{FP} = \frac{1}{2}\int_{t_0}^{t}\left(\frac{\partial\phi}{\partial t}\right)^2 dt. \tag{3.35}$$

This Fokker-Planck Lagrangian density has precisely the same form as the bulk action(3.4) with $\omega = 0$ with the identification, $t = r$. Let us evaluate S_{FP} in the classical limit. To do this, we use equation of motion from this action, which is given by

$$\frac{\partial^2\phi}{\partial t^2} = 0, \tag{3.36}$$

and the most general solution is

$$\phi = a_1 + a_2 t, \tag{3.37}$$

where a_1 and a_2 are arbitrary real constants. We will impose boundary conditions to constrain the parameters in (3.37). Suppose at a certain time t, we want to field $\phi(\tilde{t} = t) = \phi(t)$, then, the solution becomes [9]

$$\phi(\tilde{t}) = \phi(t)\frac{1 + a\tilde{t}}{1 + at}, \tag{3.38}$$

where, $a = \frac{a_2}{a_1}$. Let us plug this solution into (3.35), then we get

$$S_{FP} = \frac{1}{2}\phi(\tilde{t})\partial_{\tilde{t}}\phi(\tilde{t})\Big|_{t_0}^{t}, \tag{3.39}$$

where t_0 is initial time. *At this point, we propose that a judicious choice of the initial time t_0 precisely reproduces holographic renormalization group result. The prescription is to set*[10]

$$t_0 = -\frac{1}{a}, \tag{3.40}$$

at which the solution (3.37) becomes zero, $\phi(t_0 = -\frac{1}{a}) = 0$, and the, the range over which t varies becomes $-\frac{1}{a} < t < \infty$. Therefore, for the identification of $t = r$, we identify a subset of the interval of 't' with the interval of 'r', $0 < r < \infty$. When 'a' is positive, the stochastic

[9]This is the usual boundary condition to evaluate Fokker-Planck action. For example, see Sec.3.2.2. in [16]

[10]Our prescription for the choice of t_0 will become clear momentarily when we will discuss the Langevin dynamics

process begins before $t = 0$. In this case, we identify only a subset of the interval of t as $0 < t < \infty$ to r. For the negative value of a, the stochastic process begins after $t = 0$, then we identify entire $-\frac{1}{a} < t < \infty$ with r but then it covers only a part of the interval of 'r'. Finite non-zero value of r corresponds to UV cutoff in AdS/CFT. Thus for negative 'a' , stochastic process gives evolution of a field theory with explicit UV cutoff. With such a choice, we get

$$S_{FP} = \frac{1}{2}\frac{a}{1+at}\phi^2(t), \tag{3.41}$$

which is of the same form as (3.30) once we identify t and a with r and χ respectively. We thus see that, in this case, the prescription (2.29) is correct up to making a choice of t_0.

Langevin dynamics The Langevin equation (2.10) in this case ($S_c = 0$) becomes

$$\frac{\partial \phi}{\partial t} = \eta(t), \tag{3.42}$$

where since ϕ and η do not depend on ω, we demand

$$< \eta(t)\eta(t') >= \delta(t - t'), \tag{3.43}$$

and $< \eta(t) >= 0$. The solution of Langevin equation is given by

$$\phi(t) = \int_0^t \eta(\tilde{t})d\tilde{t} + \phi_0, \tag{3.44}$$

where ϕ_0 is an integration constant. If ϕ_0 is chosen appropriately, then (equal time)two point correlation of $\phi(t)$ will be consistent with holographic RG.

The prescription for choosing ϕ_0 is

$$\phi_0 = \int_{-\frac{1}{a}}^0 \eta(\tilde{t})d\tilde{t}. \tag{3.45}$$

We stress that we just choose initial condition for $\phi(t)$ at $t = 0$. The interval of t is still $0 < t < \infty$ for this choice. Therefore, 't' is identified with 'r'. However, once we plug (3.45) into (3.44), it has a form

$$\phi(t) = \int_{-\frac{1}{a}}^t \eta(\tilde{t})d\tilde{t}. \tag{3.46}$$

Again, $t = -\frac{1}{a}$ is a special point at which the general solution(3.37) vanishes, i.e., $\phi(t = -\frac{1}{a}) = 0$.

With this solution, one can compute (equal time)two point correlation function

$$< \phi(t)\phi(t) >_S = \int_{-\frac{1}{a}}^t \int_{-\frac{1}{a}}^t < \eta(t')\eta(t'') > dt'dt'' = t + \frac{1}{a}. \tag{3.47}$$

If we identify

$$\chi = a \text{ and } r = t, \tag{3.48}$$

then, (3.47) is precisely matches with (3.30) through the relation (2.38), since

$$< \phi(r)\phi(r) >_H = \left(\frac{\delta^2 S_B}{\delta\phi(r)\delta\phi(r)}\right)^{-1} = r + \frac{1}{\chi} \text{ and } \frac{\delta^2 S_c}{\delta\phi(r)\delta\phi(r)} = 0. \tag{3.49}$$

3.1.4 Stochastic quantization of the classical effective action: Non-zero frequency

Fokker-Planck action Our starting point is the classical action obtained from (3.16) using the relation $S_c = 2\Gamma(\phi)$

$$S_{cl} = \int_{-\infty}^{\infty} d\omega |\omega| \phi_\omega \phi_{-\omega}. \tag{3.50}$$

We first evaluate the Fokker-Planck Lagrangian density, which is given by

$$\begin{aligned} \mathcal{L}_{FP} &= \frac{1}{2}\left(\frac{\partial \phi_\omega}{\partial \tilde{t}}\right)\left(\frac{\partial \phi_{-\omega}}{\partial \tilde{t}}\right) + \frac{1}{8}\left(\frac{\delta S_{cl}}{\partial \phi_\omega}\right)\left(\frac{\delta S_{cl}}{\partial \phi_{-\omega}}\right) - \frac{1}{4}\left(\frac{\delta^2 S_{cl}}{\delta\phi_\omega \delta\phi_{-\omega}}\right) & (3.51)\\ &= \frac{1}{2}\dot{\phi}_\omega \dot{\phi}_{-\omega} + \frac{1}{2}\omega^2 \phi_\omega \phi_{-\omega} - \frac{1}{2}|\omega|\delta(0), & (3.52) \end{aligned}$$

where to evaluate Fokker-Planck Lagrangian density, we promote the field ϕ_ω

$$\phi_\omega \to \phi_\omega(\tilde{t}). \tag{3.53}$$

The Fokker-Planck Lagrangian density has the same form as the bulk Lagrangian density (3.4) up to a term proportional to the δ-function, which is just an infinite constant. This term is not relevant for the following discussion since it does not depend on the field ϕ. The 'dot' denotes derivative with respect to $\tilde{t}$, which is the stochastic time. We want set the range of the stochastic time as $0 \leq \tilde{t} \leq \infty$. Therefore, the stochastic system starts from $\tilde{t} = 0$ and settles in a thermal equilibrium at $\tilde{t} = \infty$.

Let us evaluate equation of motion, which is given by

$$0 = \ddot{\phi}_\omega - \omega^2 \phi_\omega. \tag{3.54}$$

The most general solution of the equation of motion is

$$\phi_\omega(t) = a_{1,\omega} \cosh(|\omega| t) + a_{2,\omega} \sinh(|\omega| t), \tag{3.55}$$

where $a_{1,\omega}$ and $a_{2,\omega}$ arbitrary frequency dependent functions.

Let us now look at boundary conditions. At certain time t, we want that $\phi_\omega(\tilde{t} = t) = \phi_\omega(t)$, then, the solution becomes

$$\phi_\omega(\tilde{t}) = \phi_\omega(t) \frac{\cosh(|\omega|\tilde{t}) + a_\omega \sinh(|\omega|\tilde{t})}{\cosh(|\omega| t) + a_\omega \sinh(|\omega| t)}, \tag{3.56}$$

where $a_\omega = \frac{a_{2,\omega}}{a_{1,\omega}}$. On substituting the solution (3.56) into Fokker-Planck action we get,

$$\begin{aligned} S_{FP} &= \int_{-\frac{1}{|\omega|}\coth^{-1}(a_\omega)}^{t} d\tilde{t} \int d\omega \left(\frac{1}{2}\dot{\phi}_\omega \dot{\phi}_{-\omega} + \frac{1}{2}\omega^2 \phi_\omega \phi_{-\omega}\right) = \frac{1}{2}\int d\omega \phi_\omega(\tilde{t}) \dot{\phi}_{-\omega}(\tilde{t}) \Big|_{\tilde{t} = -\frac{1}{|\omega|}\coth^{-1}(a_\omega)}^{\tilde{t}=t} & (3.57)\\ &= \frac{1}{2}\int d\omega |\omega| \phi_\omega(t) \phi_{-\omega}(t) \left(\frac{\sinh(|\omega| t) + a_\omega \cosh(|\omega| t)}{\cosh(|\omega| t) + a_\omega \sinh(|\omega| t)}\right), \end{aligned}$$

where for the second equality, we have used equation of motion(3.54) and the lower limit of the integration is chosen in the same fashion as in Sec.3.1.3. The Fokker-Planck action(3.57) has the same form as (3.31) with the identification that $a_\omega = \tilde{\phi}_\omega$. Before we discuss the Langevin dynamics let us look at the reality condition on a_ω and $\tilde{\phi}_\omega$. The reality conditions implies $a^\star_\omega = a_{-\omega}$ and $\tilde{\phi}^\star_\omega = \tilde{\phi}_{-\omega}$ respectively. In addition hermiticity of the Fokker-Planck Lagrangian density (3.57) implies both a_ω and $\tilde{\phi}_\omega$ are real.

The Langevin dynamics Let us now derive the Langevin equation(2.10) using classical action(3.50), which is given by

$$\frac{\partial \phi_\omega(t)}{\partial t} = -|\omega|\phi_\omega(t) + \eta_\omega(t). \tag{3.58}$$

The solution with appropriate boundary condition is

$$\phi_\omega(t) = \int_{-\frac{1}{|\omega|}\coth^{-1}(a_\omega)}^{t} d\tilde{t} e^{-|\omega|(t-\tilde{t})}\eta_\omega(\tilde{t}). \tag{3.59}$$

Now, we evaluate equal time two point correlator for scalar field using expectation values of $\eta_\omega(\tilde{t})$ given in (2.12)

$$\begin{aligned} < \phi_\omega(t)\phi_{\omega'}(t) >_S &= \int_{-\frac{\coth^{-1}(a_\omega)}{|\omega|}}^{t} d\tilde{t} \int_{-\frac{\coth^{-1}(a_\omega)}{|\omega|}}^{t} d\tilde{t}' e^{-|\omega|(t-\tilde{t})-|\omega'|(t-\tilde{t}')} < \eta_\omega(\tilde{t})\eta_{\omega'}(\tilde{t}') > \\ &= \frac{1}{2|\omega|}\left(1 - \frac{a_\omega - 1}{a_\omega + 1}e^{-2|\omega|t}\right)\delta(\omega+\omega'). \end{aligned} \tag{3.60}$$

We again see that the eq.(3.60) is consistent with (3.31) through the relation(2.38).

Using the definition of probability distribution(2.21), we get

$$P(\phi, t) = exp\left[-\frac{1}{2}\int d\omega|\omega|\left(\frac{(a_\omega+1)e^{|\omega|t}}{\cosh|\omega|t + a_\omega \sinh|\omega|t}\right)\phi_\omega(t)\phi_{-\omega}(t)\right]. \tag{3.61}$$

One can recognize that the kernel in exponent of $P(\phi, t)$ is precisely the inverse of the two point correlation $< \phi_\omega(t)\phi_{\omega'}(t') >_S$ in (3.60).

At this point we would like to make the following remark. One may suspect that $t_0 = -\frac{1}{|\omega|}\coth^{-1}(a_\omega)$ is not well defined when $|a_\omega| < 1$. However, we still assign our boundary condition using this form of t_0, and allowing it to have imaginary part when $|a_\omega| < 1$. In fact,

$$t_0 = -\frac{1}{|\omega|}\coth^{-1}(a_\omega) = -\frac{1}{|\omega|}\tanh^{-1}(a_\omega) \pm \frac{i\pi}{2|\omega|} \text{ for } |a_\omega| < 1. \tag{3.62}$$

To evaluate Fokker-Planck action (3.57) in this case, we choose an integration path in the complex $\tilde{t}$ plane and choose positive sign for the imaginary part of $\tilde{t}$. The contour is mostly along the real axis except at $\tilde{t} = \tanh^{-1}(a_\omega)$ where it goes parallel to imaginary axis from $\tilde{t} = \tanh^{-1}(a_\omega)$ to $\tilde{t} = \tanh^{-1}(a_\omega) + \frac{i\pi}{2|\omega|}$. The boundary action can then be written as

$$\begin{aligned} S_B &= \int_{-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)}^{t} d\tilde{t}\int d\omega \mathcal{L}_{FP} + \int_{-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)+\frac{i\pi}{2|\omega|}}^{-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)} d\tilde{t}\int d\omega\mathcal{L}_{FP} \\ &= \frac{1}{2}\int d\omega \phi_\omega(\tilde{t})\dot{\phi}_{-\omega}(\tilde{t})\Big|^{\tilde{t}=t} - \frac{1}{2}\int d\omega\phi_\omega(\tilde{t})\dot{\phi}_{-\omega}(\tilde{t})\Big|^{\tilde{t}=-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)} \\ &+ \frac{1}{2}\int d\omega\phi_\omega(\tilde{t})\dot{\phi}_{-\omega}(\tilde{t})\Big|_{\tilde{t}=-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)+\frac{i\pi}{2|\omega|}}^{\tilde{t}=-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)} \end{aligned} \tag{3.63}$$

Using the fact that

$$\dot{\phi}_\omega|_{\tilde{t}=-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)} = 0, \quad \text{and} \quad \phi_\omega|_{\tilde{t}=-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)+\frac{i\pi}{2|\omega|}} = 0, \tag{3.64}$$

the second and the third terms in the second equality in (3.63) vanishes. This gives precisely the same result as in (3.57). In the first term, the integration variable $\tilde{t}$ is on the real line, and so is t therefore, we will identify t in this case with the AdS radial coordinate r.

For the Langevin dynamics, we choose the same integration path

$$\begin{aligned} \phi_\omega(t) &= \int_{-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)+\frac{i\pi}{2|\omega|}}^{t} d\tilde{t}e^{-|\omega|(t-\tilde{t})}\eta_\omega(\tilde{t}) \\ &= \int_0^t d\tilde{t}e^{-|\omega|(t-\tilde{t})}\eta_\omega(\tilde{t}) + \phi_{\omega,0}e^{-|\omega|t}, \end{aligned} \tag{3.65}$$

where

$$\phi_{\omega,0} = \int_{-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)}^{0} d\tilde{t}e^{|\omega|\tilde{t}}\eta_\omega(\tilde{t}) + \int_{-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)+\frac{i\pi}{2|\omega|}}^{-\frac{1}{|\omega|}\tanh^{-1}(a_\omega)} d\tilde{t}e^{|\omega|\tilde{t}}\eta_\omega(\tilde{t}), \tag{3.66}$$

where $\phi_\omega(t=0) = \phi_{\omega,0}$, the initial value of ϕ_ω. The integration path in the first integral in (3.66) is a straight line along the real axis, whereas the path in the second integral is a straight line parallel to the imaginary axis. Again, t is real in the first integral and we identify it with the AdS radial coordinate r. Let us again check if $\phi_{\omega,0}$ satisfies the reality condition, $\phi^\star_{\omega,0} = \phi_{-\omega,0}$. On the real line, it is sufficient to show this that $\eta^\star_{\omega,0}(\tilde{t}) = \eta_{-\omega,0}(\tilde{t})$, however, it turns out that the reality condition will be satisfied along the second contour if we impose another condition $\eta_{\omega,0}(\tilde{t}) = \eta_{\omega,0}(\tilde{t} + \frac{i\pi}{|\omega|})$ in the complex $\tilde{t}$ plane. This is just a periodicity condition for η_0 along contour parallel to the imaginary line.

3.1.5 More on the initial conditions in stochastic quantization

In the previous discussion for the stochastic quantization, we have imposed the initial condition as $t = t_0$ in an ad hoc manner. Here we would like to provide rationale for making such a choice. For illustration consider computation of the holographic renormalization group for the zero frequency case. The radial flow of the double trace operator starts from a fixed point which is either $\chi = 0$ or $|\chi| = \infty$. Let us, for concreteness, concentrate on the $|\chi| = \infty$ fixed point. In this case, the boundary effective action (3.30) takes the form

$$S_B = \frac{1}{2}\frac{\phi^2}{r}. \tag{3.67}$$

We can also see that the Fokker-Planck action for the zero frequency case (3.41) can be rewritten as

$$S_{FP} = \frac{1}{2}\frac{\phi^2(\hat{t})}{\hat{t}}, \tag{3.68}$$

where $\hat{t}$ is a shifted time coordinate, $\hat{t} \equiv t + \frac{1}{a}$ (note that both the Langevin equation and the Fokker-Planck action possess time translation invariance.). Comparing the above two

expressions, one realizes that the boundary effective action S_B which starts from $|\chi| = \infty$ fixed point has the same form as the Fokker-Planck action if we replace r and χ in the holographic RG by $\hat{t}$ and a respectively. This implies that the choice of the initial time $t_0 = -\frac{1}{a}$ in case of the stochastic process becomes $\hat{t}_0 = 0$ and the stochastic system will start evolving from the fixed point $|a| = \infty$ (which, by our identification, is equivalent to $|\chi| = \infty$ fixed point in holographic renormalization group flows). There are several radial flows of the double trace operator for generic choices of the value of χ which do not start from fixed points. However, from the point of view of the stochastic quantization, all the stochastic time evolutions begin in the neighborhood of a fixed point but with different initial times. At $t = 0$ these stochastic time evolutions are not in the vicinity of any of the fixed points unless $a = 0$ or $|a| = \infty$. Identification of r is still done with the stochastic time t only for the range $0 \leq r \leq \infty$. Thus we see that for all the flows which do not start from the fixed point, stochastic time evolution starts from $t = -\frac{1}{a}$. While for $|\chi| = \infty$ the evolution begins from $t = 0$, for $\chi = 0_-$ it begins from $t = -\infty$. Notice that unlike the radial coordinate of AdS space which cannot take negative values, the stochastic time can begin with arbitrary negative values.

The above scheme for determining initial time is applicable to non-zero frequency case as well. The final remark is that when $|a_\omega| < 1$, $|\tilde{\phi}_\omega| = \infty$ fixed point will be obtained by shifting the stochastic time along the imaginary axis as well as the real axis. We have therefore chosen a complex initial time.

3.2 U(1) gauge fields in AdS_4

We start with the $U(1)$(Euclidean) gauge field action in AdS_4 space-time background

$$S_{bulk}[A] = \frac{1}{4}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}, \tag{3.69}$$

where the space-time indices μ, ν run from 1 to 4. The background metric is

$$ds^2 = \frac{dr^2 + \delta_{ij}dx^i dx^j}{r^2}, \tag{3.70}$$

where the indices $i, j..$ are defined boundary space-time coordinate, which run from 1 to 3 or $i = x, y$ and z and g is determinant of metric $g_{\mu\nu}$. $U(1)$ field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.71}$$

Like the massless scalar field action in AdS_2, this action is also Weyl invariant and admits alternative quantization. Under the Weyl rescaling of background metric, $ds^2 \rightarrow r^2 ds^2$, generic gauge field theory defined on AdS_4 gets mapped to that defined in 4-dimensional flat space-time. This space-time is only half of $\mathbb{R}^4$, because the radial coordinate in AdS space runs from 0 to ∞. Therefore, the action becomes

$$S_{bulk}[A] = \frac{1}{4}\int_{\mathbb{R}^4_+} d^4x F_{\mu\nu}F^{\mu\nu}, \tag{3.72}$$

where, the space-time indices are now contracted with $\delta_{\mu\nu}$ and $\mathbb{R}^4_+$ denotes a half of the 4-dimensional flat space.

The equations of motion from S_{bulk} are given by

$$
\begin{aligned}
0 &= \nabla^2 A_r - \partial_r \partial_i A_i, \\
0 &= (\partial_r^2 + \nabla^2) A_i - \partial_i(\partial_r A_r + \partial_j A_j),
\end{aligned}
\tag{3.73}
$$

where $\nabla^2 \equiv \sum_{j=1}^3 \partial_j \partial_j$. Solutions to these equations has already been obtained in [19, 27]. Let us briefly recall the solution A_μ in momentum space

$$
\begin{aligned}
A_{i,q}(r) &= A^T_{i,q}(r) - iq_i\phi^a_q(r), \qquad A_{r,q}(r) = \partial_r \phi_q(r), \;\; q_i A^T_{i,q}(r) = 0, \\
\text{and } A^T_{i,q}(r) &= A^{T(0)}_{i,q} \cosh(|q|r) + \frac{1}{|q|} A^{T(1)}_{i,q} \sinh(|q|r),
\end{aligned}
\tag{3.74}
$$

where q_i are components of three momentum along the boundary direction and the solution is obtained by using Fourier transform of the position space representation defined in a manner similar to (3.6) but this time with three boundary coordinates. $A^T_{i,q}$ is the transverse part of the gauge field, which is given by

$$\bar{A}^T_{i,q} = P_{ij}(q)\bar{A}_{j,q}, \tag{3.75}$$

where we define a projection operator,

$$P_{ij}(q) = \delta_{ij} - \frac{q_i q_j}{q^2}, \tag{3.76}$$

and $A^{T(0)}_{i,q}$ and $A^{T(1)}_{i,q}$ are q_i dependent transverse vector functions. ϕ^a is a gauge freedom which is not completely determined by equations of motion.

To proceed further we will use the radial gauge, namely $A_{r,q}(r) = 0$. In the radial gauge, the residual gauge freedom is obtained by restricting the gauge parameter $\phi_{r,q}(r)$ to be independent of r,

$$\phi_q(r) \to \phi_q. \tag{3.77}$$

Then by definition, $A^T_{i,q}(r)$ is gauge invariant under this residual gauge transformation. Another condition that we need to consider is regularity in the interior of bulk spacetime. For the regularity of the solutions at the Poincaré horizon, at $r = \infty$, we require that

$$A^{T(0)}_{ip} + \frac{1}{|p|} A^{T(1)}_{ip} = 0. \tag{3.78}$$

This removes the term proportional to $e^{|p|r}$ near the Poincaré horizon. Using this regularity condition we can write the solution in the following form

$$A^T_{i,p}(r) = A^{T(0)}_{ip} e^{-|p|r}. \tag{3.79}$$

Boundary on-shell action Substituting solutions (3.79) into the bulk on-shell action

$$S_{bulk}[A] = \frac{1}{2}\int d^3q A_{i,q}\partial_r A_{i,-q}. \tag{3.80}$$

we get

$$I_{os}[A] = S_{bulk}[A] = -\frac{1}{2}\int d^3q|q|A^{T(0)}_{i,q}A^{T(0)}_{i,-q}, \tag{3.81}$$

which is a manifestly gauge invariant action because it depends only on the transverse part of the gauge field. Canonical momentum of the gauge field is

$$\Pi^T_q = \frac{\delta S_{bulk}[A]}{\delta A^T_{i,q}} = -|q|A^T_{i,-q}. \tag{3.82}$$

The classical effective action $\Gamma[A]$ is then obtained by taking the Legendre transform of the on-shell action $I_{os}[A]$,

$$\Gamma[A] = -I_{os}[A] = \frac{1}{2}\int d^3q|q|A^{T(0)}_{i,q}A^{T(0)}_{i,-q}. \tag{3.83}$$

3.2.1 Holographic renormalization group flow of $U(1)$ gauge field theory

We start with the flow equation

$$\begin{aligned}\partial_\epsilon S_B(A) &= -\int_{r=\epsilon}\left[\frac{1}{2}\delta_{ij}\left(\frac{\delta S_B}{\delta A_{i,q}}\right)\left(\frac{\delta S_B}{\delta A_{j,-q}}\right) - \frac{1}{4}F_{ij,q}F_{kl,-q}\delta_{ik}\delta_{jl}\right] \\ &+ \int(-iq_i)\left(\frac{\delta S_B}{\delta A_{i,q}}\right)A_{r,q}\end{aligned} \tag{3.84}$$

in the momentum space. The holographic renormalization group computation of $U(1)$ gauge fields is pretty much similar to the massless scalar field case and is given in [4]. Therefore, we would like to comment only on the differences between them and then directly state the result. First of all, the main difference between them is existence of gauge degrees of freedom, which can be used to decompose the $U(1)$ gauge fields into transverse and longitudinal parts. As argued in [4], in both cases Dirichlet and Neumann boundary conditions can be imposed on the conformal boundary, equations involving transverse components are completely decoupled from those involving the longitudinal one in holographic Wilsonian RG computation. Moreover, we are only interested in radial flows of double trace coupling of transverse components of the gauge field. Therefore, the ansatz for S_B is given by

$$S_B(A) = \Lambda(\epsilon) + \int \frac{d^3q}{(2\pi)^3}\sqrt{\gamma}\mathcal{J}^T_i(q,\epsilon)g^{ij}A^T_{j,q} - \frac{1}{2}\frac{d^3q}{(2\pi)^3}\sqrt{\gamma}\mathcal{F}_T(q,\epsilon)g^{ij}A^T_{i,q}A^T_{j,-q}, \tag{3.85}$$

where, the superscript(also subscript in some of the later expressions) 'T' denotes *transverse*. Again, there are longitudinal parts in S_B, but they are decoupled.

Secondly, the equation and solution of the double trace coupling $\mathcal{F}_T$ are given by [11]

$$\partial_\epsilon f_i(q,\epsilon) = \frac{1}{(2\pi)^3} f_i(q,\epsilon) f_i(-q,\epsilon) - (2\pi)^3|q|^2, \tag{3.88}$$

$$f_i(q,\epsilon) = -(2\pi)^3 \frac{\Pi_i^T}{A_i^T}, \tag{3.89}$$

where $f_i(q,\epsilon)$ is given by

$$f_i(q,\epsilon)\delta^{ij} = \sqrt{\gamma} g^{ij} \mathcal{F}_T(q,\epsilon). \tag{3.90}$$

Π_T^i is conjugate momentum of A_i^T, which is given by $\Pi_T^i = \partial_r A_i^T$. To get some of above expressions, we have used the explicit form of the background metric(3.70). In the solution(3.89), the index 'i' is not summed over. Since A_i^T is transverse, if we suppose the three momentum, q_i is along x direction, then A_i^T will have two independent components A_y^T and A_z^T. In this case, the index i in the solution (3.89) is either y or z and arbitrary linear combination of these two solutions is not a solution since equation(3.88) is non-linear.

Finally, we obtain double trace part of the transverse gauge fields in the effective action S_B. To do that let us first write down the general solution of transverse gauge field from the bulk equation of motion(3.73)

$$A_{i,q}^T(r) = \mathcal{A}_{i,q}^{T(0)} \cosh(|q|r) + \mathcal{A}_{i,q}^{T(1)} \sinh(|q|r), \tag{3.91}$$

where $\mathcal{A}_{i,q}^{T(0)}$ and $\mathcal{A}_{i,q}^{T(1)}$ are arbitrary q_i dependent vector functions. Substituting this general solution in (3.88), we arrive at the radial flow of double trace part of transverse gauge field

$$S_B = \frac{1}{2}\int d^3q \left(\frac{\sinh(|q|r) + b_q\cosh(|q|r)}{\cosh(|q|r) + b_q\sinh(|q|r)}\right) A_{i,q}^T A_{i,-q}^T, \tag{3.92}$$

where b_q is a momentum dependent constant and we have only stated the double trace part of S_B.

3.2.2 The Fokker-Planck action and the Langevin dynamics of $U(1)$ gauge fields

In this section, we carry out stochastic quantization of $U(1)$ vector fields. Since the classical effective action (3.83) is comprised of transverse parts of gauge fields only, we suppress super(sub)script 'T' from now on, and assume all the fields in this section are transverse. Moreover, for the boundary fields, superscript (0) is used in the previous section, we will suppress this too and $A_{i,q}$ is just vector fields appearing in stochastic quantization.

[11] The other equations are given by

$$\partial_\epsilon \Lambda(\epsilon) = -\frac{1}{2(2\pi)^6}\int \delta^{ij} J_{i,q}^T J_{j,-q}^T d^3q, \tag{3.86}$$

$$\partial_\epsilon J_{i,q}^T = \frac{1}{(2\pi)^3} J_{i,q}^T f_{tq}, \tag{3.87}$$

where $J_{i,q}^T = \sqrt{\gamma} g^{ij} \mathcal{J}_{T,j}(q,\epsilon)$.

The Fokker-Planck action Using the definition of Fokker-Planck action (2.20) and prescription for the classical action

$$S_c = 2\Gamma[A], \tag{3.93}$$

we get

$$S_{FP} = \int_{t_0}^{t} dt \int d^3q \left[\frac{1}{2}(\partial_t A_{i,q})(\partial_t A_{i,-q}) + \frac{1}{2}|q|^2 A_{i,q}A_{i,-q} + \frac{1}{4}|q|\delta(0)\right], \tag{3.94}$$

where we have used

$$\frac{\delta S_{cl}[A]}{\delta A_{i,q}} = -2|q|A_{i,-q} \;\; \text{and} \;\; \frac{\delta^2 S_{cl}[A]}{\delta A^i_q \delta A_{i,p}} = -2|q|\delta^3(q+p). \tag{3.95}$$

The last term is an infinite constant, and does not contribute to the bulk dynamics. The second term can be manipulated as

$$|q|^2 A_{i,q}A_{i,-q} = -iq_j A_{i,q} iq_j A_{i,q} - iq_i A_{i,q} iq_j A_{j,q} = \frac{1}{2}F_{ij,q}F^{ij}_{-q}, \tag{3.96}$$

where we have used the fact that the gauge field that appears in the first equality is transverse. With this the Fokker-Planck action becomes

$$S_{FP} = \frac{1}{4}\int_{t_0}^{t} dt \int d^3q F_{\mu\nu,q}F^{\mu\nu}_{-q}. \tag{3.97}$$

This Fokker-Planck Lagrangian density has the same form as bulk Lagrangian density from which the boundary action is obtained.

To study stochastic time evolution of the action (3.94), let us derive equations of motion from it. The equation of motion is given by

$$0 = \ddot{A}_{i,q} - q^2 A_{i,q}. \tag{3.98}$$

The most general solution of the equation of motion is

$$A_{i,q}(t) = \bar{\mathcal{A}}_{i,q}\cosh(|q|t) + \tilde{\mathcal{A}}_{i,q}\sinh(|q|t), \tag{3.99}$$

where $\bar{\mathcal{A}}_{i,q}$ and $\tilde{\mathcal{A}}_{i,q}$ arbitrary vector functions of 3-momenta, q_i.

We impose the boundary condition by assuming that at certain time t, the gauge field satisfies $A_{i,q}(\tilde{t}=t) = A_{i,q}(t)$. Then, the solution becomes

$$A_{i,q}(\tilde{t}) = A_{i,q}(t)\frac{\cosh(|\omega|\tilde{t}) + \mathcal{B}_{i,q}\sinh(|\omega|\tilde{t})}{\cosh(|\omega|t) + \mathcal{B}_{i,q}\sinh(|\omega|t)}, \tag{3.100}$$

where $\mathcal{B}_{i,q} = \frac{\tilde{\mathcal{A}}_{i,q}}{\bar{\mathcal{A}}_{i,q}}$ and index i is not summed.

At this point, we stress that the Fokker-Planck action contains more degrees of freedom. Since, $A_{i,q}(t)$ has two independent degrees of freedom $A_{y,q}(t)$, $A_{z,q}(t)$(assuming the momentum q_i is along x-direction), initial conditions for $A_{y,q}(t)$ and $A_{z,q}(t)$ will be different, which are

determined by choices of $\mathcal{B}_{y,q}$ and $\mathcal{B}_{z,q}$. However, as argued in the last section, holographic renormalization group computation contains only a single constant b_q in (3.92).[12] To reproduce this correctly, we set

$$d_q \equiv \mathcal{B}_{y,q} = \mathcal{B}_{z,q}. \tag{3.101}$$

After this, we plug the solution (3.100) into Fokker-Planck action and evaluate it. It is given by

$$\begin{aligned} S_{FP} &= \int_{-\frac{1}{|q|}coth^{-1}(d_q)}^{t} d\tilde{t} \int d^3q \left(\frac{1}{2}(\partial_t A_{i,q})(\partial_t A_{i,-q}) + \frac{1}{2}|q|^2 A_{i,q}A_{i,-q}\right) \\ &= \frac{1}{2}\int d^3q A_{i,q}(\tilde{t})\dot{A}_{i,-q}(\tilde{t})\Big|_{\tilde{t}=-\frac{1}{|q|}\coth^{-1}(d_q)}^{\tilde{t}=t} \\ &= \frac{1}{2}\int d^3q|q|A_{i,q}(t)A_{i,-q}(t)\left(\frac{\sinh(|q|t)+d_q\cosh(|q|t)}{\cosh(|q|t)+d_q\sinh(|q|t)}\right), \end{aligned} \tag{3.102}$$

where for the second equality, we have used equation of motion(3.98) and the lower limit of the integration has chosen by the same way that we have done in Sec.3.1.3. The Fokker-Planck action(3.102) is the same form with (3.92) under the condition that $d_q = b_q$.

The Langevin dynamics We start with Langevin equation from the classical action (3.93)

$$\frac{\partial A_{i,q}(t)}{\partial t} = -\frac{\delta S_{cl}[A_{i,q}(t)]}{\delta A_{i,-q}(t)} + \eta_{i,q}(t) = -|q|A_{i,q} + \eta_{i,q}(t). \tag{3.103}$$

The solution of this equation with the initial condition prescribed in Sec.3.1.3 is

$$A^a_{i,q}(t) = \int_{-\frac{1}{|q|}\coth^{-1}(\mathcal{B}_{i,q})}^{t} d\tilde{t}e^{-|q|(t-\tilde{t})}\delta_{ij}\eta_{j,q}(\tilde{t}) \tag{3.104}$$

where the index j is summed over but index i is free.

The same feature of stochastic quantization arises here too. For each component of gauge fields, one can assign different boundary conditions by choosing $\mathcal{B}_{i,q}$ differently. However, to reproduce holographic renormalization group calculations, we impose the same condition as (3.101).

With such a choice, we compute stochastic correlation function using

$$< \eta_{i,q}(t)\eta_{j,q'}(t') >= \delta_{ij}\delta^3(q-q')\delta(t-t'), \tag{3.105}$$

which is given by

$$< A_{i,q}(t)A_{j,q'}(t) >= \delta_{ij}\delta^3(q-q')\frac{1}{2|q|}\left(1 - \frac{d_q-1}{d_q+1}e^{-2|q|t}\right). \tag{3.106}$$

(3.106) is consistent with (3.92) through the relation(2.38), provided that $d_q = b_q$ and $t = r$.

[12] This distinction occurs because the double trace coupling in the holographic renormalization group satisfies a non-linear equation (3.88) but there is no such obvious condition appearing in the Fokker-Planck, and as we will see later, in the Langevin dynamics as well. It would be useful to understand this issue better to develop a closer analogy between these two formalisms.

4 Conclusion and Open Questions

In this paper, we have shown that in the case that the bulk action is Weyl invariant and allows alternative quantization, stochastic quantization of the classical action which is given by $S_c = 2\Gamma$, where Γ is classical effective action from the bulk gravity theory without any deformations precisely captures the radial flow of double trace deformation coupling in holographic Wilsonian renormalization group computation. We have studied this proposal by analyzing a couple of examples, (minimally coupled)massless scalar field in AdS_2 and $U(1)$ gauge fields in AdS_4 as bulk theories. In these examples, the radial flow of the double trace couplings is precisely obtained from the stochastic time evolution of the corresponding Fokker-Planck action and Langevin dynamics.

Even if these examples are quite successful, there are many open questions, some of which we will list here.

- In our example, we only dealt with Weyl invariant action. If the bulk action is not Weyl invariant, there must be divergent pieces in the near boundary expansion of the bulk solutions. In such cases, one needs to add counter-terms to cancel divergent contributions to the boundary on-shell action. These counter-terms could modify the identification, $S_c = 2\Gamma$.

- Not many examples of interacting boundary conformal field theories have been studied either in the holographic Wilsonian renormalization group method or in the stochastic quantization method. The Langevin dynamics, however, does provide a method to deal with interactions in perturbative expansion in small coupling[16]. Nevertheless, application of this method to study the relation between holographic Wilsonian RG and stochastic quantization is still an open question. In [27], the authors developed boundary theories of $SU(2)$ Yang-Mills in AdS_4 and which provide boundary effective action with exotic momentum dependent interaction vertices. This is a natural extension of $U(1)$ theory in AdS_4 to add interactions in it and at the same time retains some of the merits of the $U(1)$ case: the bulk action is still Weyl invariant and allows alternative quantization. One might think about stochastic quantization of this boundary theory to extend our argument further.

- The last question is how the relation will be modified if the bulk geometry is not pure AdS space. For example, in [4], the authors study holographic Wilsonian RG in extremal black brane background. In this case, there are emergent IR-CFT near black brane horizon since the near horizon geometry is AdS_2 and there will be more than one (non-trivial) IR fixed point. The question is whether stochastic quantization can capture these fixed points appropriately.

- For non-extremal black brane case, we have to deal with conformal field theories at finite temperature. Stochastic noise does not provide a notion of 'temperature' in the sense that it does not correspond to black brane temperature. Even if there is stochastic noise, in our examples the corresponding bulk geometry is still pure AdS. It will therefore be interesting to figure out how a finite temperature system from AdS/CFT would

be accommodated in our prescription. For some realted literature, see [30]. A better understanding of this will put this proposal on a firmer footing.

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